JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 3, 117–121 (1961)

Notes on Discrete Potential Theory

JOHN G. KEMENY AND J. LAURIE SNELL*

Department of Mathematics, Dartmouth College, Hanover, New Hampshire

In [2] we have shown that the matrices

$$C_{ij} = \operatorname*{Lim}_{n} [N_{jj}^{(n)} - N_{ij}^{(n)}] \qquad \text{and} \qquad G_{ij} = \operatorname*{Lim}_{n} [N_{ii}^{(n)} \cdot \alpha_j/\alpha_i - N_{ij}^{(n)}]$$

serve as dual potential operators for potential measures (row vectors) and potential functions (column vectors), respectively, for recurrent Markov chains. Here $N_{ij}^{(n)}$ is the mean number of times in state j in the first n steps, if the process starts in i, and α is a positive stationary measure. (Such a measure is unique up to a constant factor, and only the ratio of two components enters our formula.) While these limits always exist for positive recurrent chains, Orey has shown in [3] that they need not exist for null chains.

Specifically, Orey shows that the Abel-summability of C_{01} is equivalent to the existence of

$$\operatorname*{Lim}_{t \to 1} \frac{1 - F(t)}{1 - G(t)},$$

where $F(t)$ is the generating function of the first return times from 0 to the set $\{0, 1\}$, while $G(t)$ generates the return times from 1 to $\{0, 1\}$. From this he concludes that there must be null chains for which the limits are not even Abel-summable.

Actually, it is easy to construct a concrete example of such a chain. It will be the combination of two rooted trees, having 0 and 1 as roots, respectively. From 0 we enter the kth branch of the first tree with probability $1/2^k$, $k = 1, 2, \ldots$. The kth branch has $n_{2k-1} - 1$ states on it, and once it is entered, the process moves through these states deterministically. At the end of the branch the process moves to state 1. From 1 it proceeds in a similar manner to the second tree, only here the

* The preparation of this manuscript was aided by the National Science Foundation through a grant to the Dartmouth Mathematics Project.

kth branch has $n_{2k} - 1$ states, and at the end it moves to state 0. It is obvious from the construction that $F(t) = \Sigma_k 1/2^k \cdot t^{n_{2k-1}}$ and $G(t) = \Sigma_k 1/2^k \cdot t^{n_{2k}}$.

Then we need simply choose the exponents n_k and a sequence of numbers ε_k so that $\varepsilon_k n_k \to \infty$ while $2^{k/2} \varepsilon_k n_{k-1} \to 0$. (For example, $n_k = 2^{k^2}$ and $\varepsilon_k = 2^{-k(k-1)}$ will do.) Then the ratio $[1 - F(t)]/[1 - G(t)]$ will fail to converge along the sequence $t_k = 1 - \varepsilon_k$. Indeed, on the subsequence of even k it tends to 1/2, while for odd k it tends to 1. Thus C_{01} does not exist and is not even Abel-summable for this chain.

Even if the limits do not exist, certain combinations may. In [2] we proved under a weak hypothesis that the limit defining $C_{ij} + \alpha_j/\alpha_i \cdot C_{ji}$ exists even if C_{ij} does not. Dually, $G_{ij} + \alpha_j/\alpha_i \cdot G_{ji}$ exists under the same conditions, and either limit is ${}^iN_{jj}$, the mean number of times the chain is in state j, starting at j, before reaching state i. Orey in [3] showed that these sequences are always Abel-summable.

Chung has shown that these limits exist for a large class of chains (see [1]). Actually, a slight modification of his method will establish that the limits exist for *all* null recurrent chains.

In his proof Chung makes use of the following lemma:

If $\Sigma_\nu |a_\nu| < \infty$, $|b_n| \leqslant A < \infty$, *and* $\lim_n (b_n - b_{n-1}) = 0$, *then* $\lim_n \Sigma_{\nu=1}^n |a_\nu| \, |b_n - b_{n-\nu}| = 0$.

We showed in [2] that even though the sequences defining C_{ij} and G_{ij} may not converge, they are positive and bounded by ${}^iN_{jj}$. Thus we obtain for the sequence defining $\alpha_j/\alpha_i \cdot G_{ji}$ that

$$0 \leqslant N_{jj}^{(n)} - \alpha_j/\alpha_i \cdot N_{ji}^{(n)} \leqslant \alpha_j/\alpha_i \cdot {}^jN_{ii} = {}^iN_{jj}.$$

Let $f_{ij}^{(\nu)}$ be the probability, starting at i, that state j is reached for the first time on the νth step. Following Chung's proof we have

$$N_{ji}^{(n)} = \sum_{\nu=1}^{n} f_{ij}^{(\nu)} N_{ji}^{(n)} + \sum_{n+1}^{\infty} f_{ij}^{(\nu)} N_{ji}^{(n)},$$

$$N_{jj}^{(n)} = \sum_{\nu=1}^{n} f_{ij}^{(\nu)} N_{jj}^{(n)} + \sum_{\nu=n+1}^{\infty} f_{ij}^{(\nu)} N_{jj}^{(n)},$$

$$N_{ij}^{(n)} = \sum_{\nu=1}^{n} f_{ij}^{(\nu)} N_{jj}^{(n-\nu)},$$

$$N_{ii}^{(n)} = \sum_{\nu=1}^{n} f_{ij}^{(\nu)} N_{ji}^{(n-\nu)} + {}^jN_{ii}^{(n)}.$$

The first two are trivial; the last two are clear probabilistically. Multiplying these equations in order by $-\alpha_j/\alpha_i$, 1, -1, and α_j/α_i, and adding, we obtain

$$N_{jj}^{(n)} - N_{ij}^{(n)} + \frac{\alpha_j}{\alpha_i}(N_{ii}^{(n)} - N_{ji}^{(n)}) = \sum_{\nu=1}^{n} f_{ij}^{(\nu)}\left[N_{jj}^{(n-\nu+1,n)} - \frac{\alpha_j}{\alpha_i}N_{ji}^{(n-\nu+1,n)}\right] +$$
$$\sum_{\nu=n+1}^{\infty} f_{ij}^{(\nu)}\left[N_{jj}^{(n)} - \frac{\alpha_j}{\alpha_i}N_{ji}^{(n)}\right] + \left(\frac{\alpha_j}{\alpha_i}\right){}^{j}N_{ii}^{(n)}. \quad (1)$$

Let $b_n = N_{jj}^{(n)} - \alpha_j/\alpha_i\, N_{ji}^{(n)}$. Since these are bounded, the second term in (1) approaches zero. Also for any null chain $P_{ij}^{(n)}$ approaches zero. Hence $(b_n - b_{n-1}) \to 0$. Thus, by the lemma, the first term in (1) also approaches 0. Finally, since $(\alpha_j/\alpha_i){}^{j}N_{ii} = {}^{i}N_{jj}$ we have:

THEOREM 1. *For any null-recurrent Markov chain*

$$\lim_n\left(N_{jj}^{(n)} - N_{ij}^{(n)} + \frac{\alpha_j}{\alpha_i}(N_{ii}^{(n)} - N_{ji}^{(n)})\right) = {}^{i}N_{jj}.$$

This establishes our basic result that the sequences defining $C_{ij} + \alpha_j/\alpha_i\, C_{ji}$ always converge. For aperiodic positive recurrent chains we have the stronger result that C itself always exists.

In [2] we showed that the existence of G_{01} is equivalent to the convergence of the sequence $\Sigma_k\, P_{ak}^{(n)}\, {}^0H_{k1}$, where ${}^0H_{k1}$ is the probability, starting at k, of hitting 1 before 0. We considered a weakened version of this condition, namely that

$$\sum_k (P_{ak}^{(n)} - P_{bk}^{(n)})\,{}^0H_{k1} \quad \text{converges to } 0. \qquad (*)$$

We now know that G_{01} may not even be Abel-summable, hence the same is true for the equivalent condition. However, the weaker condition (*) may still be universally true. (This is the "weak condition" from which we deduced the existence of $C_{ij} + \alpha_j/\alpha_i \cdot C_{ji}$ in [2].) We will now show that the sequence in (*) is always at least C_1-summable to 0.

LEMMA. *For any recurrent chain,* $(1/n)\,\Sigma_{\nu=1}^{n}\, \nu f_{ab}^{(\nu)} \to 0$.

PROOF: Let $H_{ab}^{(n)} = \Sigma_{\nu=1}^{n} f_{ab}^{(\nu)}$. This sequence converges to 1. Hence we also have limit 1 for

$$\frac{1}{n}\sum_{k=1}^{n} H_{ab}^{(k)} = \frac{1}{n}\sum_{\nu=1}^{n}(n+1-\nu)f_{ab}^{(\nu)} = \frac{n+1}{n}H_{ab}^{(n)} - \frac{1}{n}\sum_{\nu=1}^{n}\nu f_{ab}^{(\nu)}.$$

But the first term on the right tends to 1, hence the second tends to 0, and the lemma follows.

THEOREM 2.

$$\operatorname{Lim}_n \frac{1}{n} \sum_k [N_{ak}^{(n)} - N_{bk}^{(n)}]\, {}^0H_{k1} = 0.$$

PROOF: Let us start the process at state a. Let us define a state to be "favorable" if from it we enter 1 before 0. Then $1/n \,\Sigma_k N_{ak}^{(n)}\, {}^0H_{k1}$ is the mean of the average number of favorable states entered in the first n steps. The other term may be thought of as the mean of the average number of favorable states entered in the n steps after b is reached. If b is reached on step t, then the difference counts the number of favorable states in the first t steps, minus the number between n and $n + t$. Either the positive or negative term is no greater than $(1/n)M_a[\min(n, t)]$. But this is

$$\frac{1}{n}\left[\sum_{\nu=1}^{n} \nu f_{ab}^{(\nu)} + \sum_{\nu>n} n f_{ab}^{(\nu)}\right].$$

Since $\Sigma_\nu f_{ab}^{(\nu)}$ converges, the second term tends to 0. By the lemma, the first term also goes to 0. Hence the Theorem follows.

As we indicated above, the existence of the limits ${}^0\lambda_i = \lim_n \Sigma_k P_{ak}^{(n)}\, {}^0H_{ki}$ is equivalent to the existence of G. Such chains are called *normal*. A natural generalization is $\lambda_i^E = \lim_n \Sigma_k P_{ak}^{(n)} B_{ki}^E$; B_{ki}^E is the probability starting at k that the set of states E is entered at i. Thus ${}^0\lambda_i = \lambda_i^{\{0,1\}}$. We showed in [2] that for a normal chain λ^E exists for all finite E, and under an appropriate restriction exists for all E. We left open the question of whether λ^E ever fails to exist in a normal chain. We can now produce such examples.

The basic class of examples in [2] is the following: The states are the natural numbers. From a state k the chain either moves to $(k + 1)$, with probability p_{k+1}, or to 0. The chain is recurrent if and only if $\lim_{y\to\infty} {}^0H_{xy} = 0$ for each x. We will show that if an example of this type is null, then there always exists a set for which λ^E fails to converge.

Let us choose an ε, $0 < \varepsilon < 1/3$. We construct a sequence of numbers n_k and of disjoint finite sets E_k, so that $S_k = \bigcup_{i=0}^{k} E_i$ is a set of consecutive integers. Let $n_0 = 0$, and $E_0 = \{0\}$, then $P_{0E_0}^{(n_0)} = 1$. Let n_{k+1} be any number sufficiently large so that $P_{0S_k}^{(n_k+1)} < \varepsilon$. This can always be achieved since the set S_k is finite and the chain is null.

We will choose E_{k+1} of the form $E_k = \{0, 1, \ldots, a_{k+1}\} - S_k$, and $a_{k+1} \geqslant n_{k+1}$. Since the process can move only one step to the right, $P_{0S_{k+1}}^{(n_k+1)} = 1$, and hence $P_{0E_{k+1}}^{(n_k+1)} > 1 - \varepsilon$. By induction, for all k, $P_{0E_k}^{(n_k)} > 1 - \varepsilon$. If $(k+1)$ is odd, we let $a_{k+1} = n_{k+1}$. If $(k+1)$ is even, we choose a_{k+1} large enough so that ${}^0H_{n_{k+1}a_{k+1}} < \varepsilon$. As remarked above, this is always possible for a recurrent example.

Finally, we let $E = \{0\} \cup E_1 \cup E_3 \cup E_5 \cup \ldots$.

$$\sum_{i \in E_k} P_{0i}^{(n_k)} B_{i0}^E \leqslant \sum_i P_{0i}^{(n_k)} B_{i0}^E \leqslant \varepsilon + \sum_{i \in E_k} P_{0i}^{(n_k)} B_{i0}^E.$$

If k is odd and $i \in E_k$, then $B_{ii}^E = 1$, hence $B_{i0}^E = 0$. Thus the upper bound is ε for all odd k. If k is even and $i \in E_k$ and $P_{0i}^{(n_k)} > 0$, then $i \leqslant n_k$ and $B_{i0}^E > 1 - \varepsilon$. Thus the lower bound is at least $(1-\varepsilon)P_{0E_k}^{(n_k)} > 1 - 2\varepsilon$ for all even k. Hence $\lambda_0{}^E = \lim_n \Sigma_i P_{0i}^{(n)} B_{i0}^E$ does not exist.

References

1. Chung, K. L. Some remarks on taboo probabilities. *Illinois J. Math.*
2. Kemeny, J. G., and Snell, J. L. Potentials for denumerable Markov chains. *J. Math. Anal. and Appl.* in press.
3. Orey, S. Sum arising in the theory of Markov chains. *Proc. Am. Math. Soc.*